

A SMALL PARAMETER APPROACH FOR FEW-BODY PROBLEMS

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Abstract

A procedure to solve few-body problems is developed which is based on an expansion over a small parameter. The parameter is the ratio of potential energy to kinetic energy for states having not small hyperspherical quantum numbers, $K > K_0$. Dynamic equations are reduced perturbatively to equations in the finite-dimension subspace with $K \leq K_0$. Contributions from states with $K > K_0$ are taken into account in a closed form, i.e. without an expansion over basis functions. Estimates on efficiency of the approach are presented.

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I. INTRODUCTION

While the nuclear force cannot be treated as a perturbation, the potential energy of, say, a bound state of a nucleus is in fact comparable to its kinetic energy only for a limited number of its components. These are components having low values of the hypermomentum K . For all remaining components kinetic energy exceeds potential energy which allows solving few-nucleon problems perturbatively.

This approach is suggested below and is a development of that of Ref. [1] where a perturbation method has been given to solve large systems of bound-state linear equations pertaining to the hyperspherical-hyperradial expansion.[20] The method proved to be efficient [3, 4]. The expansion parameter was the potential-to-kinetic-energy matrix element ratio. However, at $A>3$ it is the calculation of matrix elements themselves that requires a massive computational effort.

The difficulty stems from a swift rise, as K increases, of the number of hyperspherical harmonics having the same K . (The larger is a number of particles the swifter is the rise.) Selection of hyperspherical harmonics to reduce the computational effort, see [1, 5, 6], is not efficient for $A>4$ bound states and is not justified in reaction calculations. In the method given below the difficulty is removed. No expansions are employed when K is not small.

Recently a considerable progress in methods for solving few-body problems has been achieved. However, those developments have limitations not arising in the presented method. In particular, the well-known Green Function Monte Carlo (GFMC) method is the method to calculate a bound state of a system, and it does not suit to calculate reactions. (Although the simplest scattering problems may be considered in its frames.) Unlike this method, the method presented below is suitable for calculating reactions of a general type. Besides, the GFMC method is not convenient in that providing separate observables it does not provide the wave function of a bound state as an outcome of the calculation. And in the framework of the method described below bound state wave functions are calculated in a rather simple form suitable for subsequent applications.

Recently a way was found to extend the Faddeev-Yakubovsky $A=4$ calculations over the energy range above the four-body breakup threshold [7]. However, Yakubovsky type calculations require too much numerical effort. The amount of calculations is less in the scheme below.

At solving few-body problems, convergence of results for calculated observables was accelerated with the help of the effective interaction methods. Such methods were developed in the framework of the oscillator expansion [8] and the hyperspherical expansion [9]. In this approach, a true Hamiltonian is replaced with some effective Hamiltonian acting in a subspace of only low excitations. When, formally, the latter subspace is enlarged an effective Hamiltonian turns to a true one. An effective Hamiltonian is constructed from a requirement that its ingredients, as defined in the subspace of low excitations, reproduce some properties of the corresponding ingredients of a true Hamiltonian in the total space. It has been shown [8, 9] that this, indeed, leads to improvement of convergence for observables considered.

Higher excitations are disregarded in such type calculations. It is clear, however, that correlation effects related to higher excitations cannot be reproduced by any state vector lying in the allowed subspace of only low excitations. Consider e.g. the mean value, $\langle \Psi_0 | H | \Psi_0 \rangle$, of such an "observable" as a true Hamiltonian. It follows from the variational principle that an approximate state Ψ_0 obtained with such a method provides poorer approximation to the true $\langle \Psi_0 | H | \Psi_0 \rangle$ value than Ψ_0 given by the simple diagonalization of a Hamiltonian in the same subspace of low excitations. But even the latter $\langle \Psi_0 | H | \Psi_0 \rangle$ value is a very poor approximation for realistic Hamiltonians. On the contrary, the method given below provides an approximate state vector that is apparently close to a true state vector both as to its low excitation component and to its high excitation component.

And speaking of reaction calculations in the framework of the dynamic schemes employed below one should also take into account that a rate of convergence is determined not only by the properties of a Hamiltonian but also by those of the source-term q entering the equations. But these properties are apparently ignored at constructing effective Hamiltonians. Unlike this, the method developed below provides state vectors genuinely close to the true ones both for bound state problems and reaction problems.

Development of efficient microscopic methods for nuclear physics is timely now because of a necessity to test nuclear forces derived from the effective field theory.

The method is given in the next section, and its implementation is considered in Sec. 3. In Sec. 4. estimates on its efficiency are presented. In particular, the method seems to be promising to test realistic nuclear forces in four-nucleon reaction problems and in scattering problems below three-body breakup thresholds in the ^5He , ^7Li , ^7Be , and ^8Be systems. This opens up basically a new field.

II. THE SMALL PARAMETER EXPANSION

We consider first the bound state problem

$$(H - E_i)\Psi_i = 0, \quad (1)$$

where $H = T + V$ is an A -body Hamiltonian. We split the whole space of states into the subspaces with $K \leq K_0$ and $K > K_0$ and we denote Ψ_i^l and Ψ_i^h the components of the solution Ψ_i that lie, respectively, in these subspaces. Let us denote P_{K_0} and Q_{K_0} projectors onto the $K \leq K_0$ and $K > K_0$ subspaces, respectively. Let $E_i^{(0)}$ be the zero approximation eigenvalue that arises when coupling to the $K > K_0$ subspace is disregarded,

$$P_{K_0} (H - E_i^{(0)}) \Psi_i^{l(0)} = 0. \quad (2)$$

The state $\Psi_i^{l(0)}$ is the corresponding zero approximation eigenfunction.

Let us also define a Green function in the $K > K_0$ subspace. The operators of hyper-rotations commute with the kinetic energy operator. Therefore, if Ψ_Q is a state lying in the $K > K_0$ subspace then $T\Psi_Q$ also belongs to this subspace. We shall consider also the "correction operators" δH specified below that have the same property. Then

$$\varphi_Q = (T + \delta H - E_i^{(0)})\Psi_Q \quad (3)$$

is a state belonging to the $K > K_0$ subspace. We also assume that $T + \delta H$ possess only continuum spectrum. Then for any state f_Q in the $K > K_0$ subspace one has $(T + \delta H - E_i^{(0)})f_Q \neq 0$. Taking also into account that the operator from (3) possess a complete set of eigenstates with $K > K_0$ one concludes that there exists a unique state Ψ_Q corresponding to any given φ_Q in (3). Thus we can define the corresponding Green function

$$G_{K_0} = \left(T + \delta H - E_i^{(0)} \right)^{-1} Q_{K_0}. \quad (4)$$

Taking into account that $P_{K_0} T Q_{K_0} = Q_{K_0} T P_{K_0} = 0$ we write down Eq. (1) in the form

$$P_{K_0} (H - E_i) \Psi_i^l = -P_{K_0} V \Psi_i^h, \quad (5)$$

$$\Psi_i^h = -G_{K_0} [V \Psi_i^l + (U - \Delta E_i) \Psi_i^h], \quad (6)$$

where

$$U = V - \delta H, \quad \Delta E_i = E_i - E_i^{(0)}.$$

When the hyperangular momentum K_0 is sufficiently large kinetic energies of states in the $K > K_0$ subspace are high. Then in accordance with Eq. (4) Ψ_i^h is "small", so that one may treat the second term in the right-hand side of Eq. (6) as a perturbation. Thus one may express the component Ψ_i^h in terms of Ψ_i^l perturbatively and obtain dynamic equations for the latter component alone. Namely, one may write

$$\Psi_i^h = -\Gamma V \Psi_i^l, \quad (7)$$

where

$$\Gamma = G_{K_0} - G_{K_0}(U - \Delta E_i)G_{K_0} + G_{K_0}(U - \Delta E_i)G_{K_0}(U - \Delta E_i)G_{K_0} - \dots \quad (8)$$

(We mention that, say, the contribution from $\Delta E_i G_{K_0}^2$ here is not of the same magnitude as that from $G_{K_0} U G_{K_0}$.) The dynamic equation (1) may then be put in the form of an equation in the subspace of low excitations only,

$$P_{K_0}(T + V - V\Gamma V - E_i)\Psi_i^l = 0. \quad (9)$$

The quantity $-V\Gamma V$ represents a genuine effective interaction that reproduces the effect of the excluded $K > K_0$ subspace.

It is convenient to treat perturbatively not only subsequent contributions to the Γ operator but also the whole $-V\Gamma V$ interaction. This leads to an expansion over powers of both $G_{K_0}U$ and $G_{K_0}V$. We replace G_{K_0} with λG_{K_0} and we seek for $\Psi_i^l(\lambda)$ as an expansion,

$$\Psi_i^l(\lambda) = \sum_{m \geq 0} \lambda^m \Psi_i^{l(m)}, \quad E_i(\lambda) = \sum_{m \geq 0} \lambda^m E_i^{(m)} \quad (10)$$

setting $\lambda = 1$ at the end. The zero order eigenstates and eigenvalues are given by Eq. (2). Equations for higher order corrections are of the form

$$P_{K_0}(H - E_i^{(0)})\Psi_i^{l(m)} = q^{(m)} \quad (11)$$

with a given source term. Explicit form of the equations for $\Psi_i^{l(1)}$ and $\Psi_i^{l(2)}$ is

$$P_{K_0}(H - E_i^{(0)})\Psi_i^{l(1)} = E_i^{(1)}\Psi_i^{l(0)} + P_{K_0}VG_{K_0}V\Psi_i^{l(0)}, \quad (12)$$

$$\begin{aligned} P_{K_0}(H - E_i^{(0)})\Psi_i^{l(2)} = & E_i^{(1)}\Psi_i^{l(1)} + E_i^{(2)}\Psi_i^{l(0)} + P_{K_0}VG_{K_0}V\Psi_i^{l(1)} \\ & - P_{K_0}VG_{K_0}UG_{K_0}V\Psi_i^{l(0)}. \end{aligned} \quad (13)$$

(The term from (8) with $\Delta E_i \rightarrow E_i^{(1)}$ would first appear at $m = 3$.)

Obviously, solutions to Eqs. (11) are not unique: at given lower order corrections entering $q^{(m)}$ the solution $\Psi_i^{l(m)}$ is determined up to $\text{const} \cdot \Psi_i^{l(0)}$. It is convenient to select unique solutions imposing e.g. the usual perturbation theory condition

$$\langle \Psi_i^{l(0)} | \Psi_i^l(\lambda) \rangle = \langle \Psi_i^{l(0)} | \Psi_i^{l(0)} \rangle. \quad (14)$$

This is a normalization condition since up to a numerical factor $\Psi_i^l(\lambda)$ are determined by Eq. (1) type equations. According to the first of Eqs. (10) this is equivalent to the set of conditions

$$\langle \Psi_i^{l(0)} | \Psi_i^{l(m)} \rangle = 0, \quad m = 1, 2, \dots \quad (15)$$

(Use of any other condition of Eq. (14) type whose right-hand side includes additional terms tending to zero at $\lambda \rightarrow 0$ would lead merely to a different normalization of the sum of $\Psi_i^{l(m)}$.)

In order Eqs. (11) be self-consistent the conditions $\langle \Psi_i^{l(0)} | q^{(m)} \rangle = 0$ should be fulfilled. This determines the energy corrections $E_i^{(m)}$. (In the present consideration complications that arise in cases of possible non-trivial (quasi) degeneracies of levels are disregarded.) We get, in particular,

$$E_i^{(1)} = -\frac{\langle \Psi_i^{l(0)} | V G_{K_0} V | \Psi_i^{l(0)} \rangle}{\langle \Psi_i^{l(0)} | \Psi_i^{l(0)} \rangle}, \quad (16)$$

$$E_i^{(2)} = \frac{\langle \Psi_i^{l(0)} | V G_{K_0} U G_{K_0} V | \Psi_i^{l(0)} \rangle - \langle \Psi_i^{l(0)} | V G_{K_0} V | \Psi_i^{l(1)} \rangle}{\langle \Psi_i^{l(0)} | \Psi_i^{l(0)} \rangle}. \quad (17)$$

Since by construction G_{K_0} is positive definite the correction $E_i^{(1)}$ is negative. (Besides, one has in (17) $\langle \Psi_i^{l(0)} | V G_{K_0} V | \Psi_i^{l(1)} \rangle = -\langle \Psi_i^{l(1)} | H - E_i^{(0)} | \Psi_i^{l(1)} \rangle$.)

Thus we have dynamic equations of the Eqs. (10), (11) form which do not involve excitations with $K > K_0$ at all. This is an advantage since the $K \leq K_0$ subspace spans only a finite, hopefully not too large, number of HH.

The complementary $K > K_0$ component of a state sought for is given by Eq. (7). This can be represented as

$$\Psi_i^h = \sum_{m \geq 1} \Psi_i^{h(m)}, \quad (18)$$

where, in particular,

$$\Psi_i^{h(1)} = -G_{K_0} V \Psi_i^{l(0)}, \quad (19)$$

$$\Psi_i^{h(2)} = -G_{K_0} V \Psi_i^{l(1)} + G_{K_0} U G_{K_0} V \Psi_i^{l(0)}. \quad (20)$$

The low excitation component Ψ_i^l is obtained above in the form of a hyperspherical expansion. This component may be stored in this form for use in the relations of the Eqs. (19), (20) type and also in other applications. The complementary high-excitation component Ψ_i^h is reconstructed as a quadrature and may be applied in such a form. This component does not involve any expansion.

According to the variational principle the energy eigenvalue calculated as the average value of a Hamiltonian over $\Psi_i^{l(0)} + \Psi_i^{l(1)} + \Psi_i^{h(1)}$ is accurate up to the order $m = 3$.

Let us also comment on applying of a such type expansion for calculation of reactions. First we shall proceed in the framework of the following well-known procedure [10, 11]. Consider reactions below the three-fragment breakup threshold when only two-fragment channels are open. Let Ψ_i be a continuum spectrum state and denote N the number of open two-fragment channels. The ansatz

$$\Psi_i = \phi_i^{(1)} + \sum_{j=1}^N f_{ij} \phi_j^{(2)} + X, \quad (21)$$

is used where $\phi_i^{(1)}$ and $\phi_j^{(2)}$ represent the "channel" states of two possible types, f_{ij} are reaction amplitudes to be determined, while X is localized and is sought for as an expansion over hyperspherical harmonics. In the three-nucleon case the procedure is applicable also above the three-nucleon breakup threshold. In this case X describes breakup to free particles at large distances. The equation determining X is

$$(H - E)X = q, \quad (22)$$

$$q = -\bar{\phi}_i^{(1)} + \sum_{j=1}^N f_{ij} \bar{\phi}_j^{(2)},$$

where $\bar{\phi}_i^{(1),(2)} = (H - E)\phi_i^{(1),(2)}$ are localized states. The state X is found from Eq. (22) up to reaction amplitudes f_{ij} (cf. below),

$$X = X_i + \sum_{j=1}^N f_{ij} X_j, \quad (23)$$

and f_{ij} are obtained from N additional linear equations.

Proceeding as above we represent X as $X^l + X^h$. We have

$$P_{K_0} (T + V - VT V - E) X^l = P_{K_0} (q - VT q), \quad (24)$$

$$X^h = -\Gamma (VX^l - q) . \quad (25)$$

Here Γ is of the form of Eq. (8) with ΔE_i being omitted and with $E_i \rightarrow E$ in Eq. (4). Writing as above $X^l = \sum_m X^{l(m)}$, $X^h = \sum_m X^{h(m)}$ one obtains equations for $X^{l(m)}$ and expressions for $X^{h(m)}$ performing an expansion over G_{K_0} . One has, in particular,

$$P_{K_0}(H - E)X^{l(0)} = P_{K_0}q \quad (26)$$

while the corresponding equations for $X^{l(1),(2)}$ and expressions for $X^{h(1),(2)}$ are obtained from Eqs. (12), (13) and (19), (20) with $E_i^{(1),(2)}$ being omitted and with the replacements $E_i^{(0)} \rightarrow E$, and $\Psi^{l,h(1),(2)} \rightarrow X^{l,h(1),(2)}$, $V\Psi^{l(0)} \rightarrow VX^{l(0)} - q$.

One may note that Eq. (22) has a localized solution only when the amplitudes f_{ij} equal to their true values. Otherwise, the solution includes an admixture of cluster components. On the contrary, if a finite number of terms is retained in the expansion of Γ over powers of G_{K_0} then the localized solution exists at any f_{ij} allowing the representation (23). The reason is that the role of higher terms in the expansion of Γ increases at large distances. When f_{ij} in (22) not coincide with their true values these higher terms are responsible for description of the large distance clusterization (cf. below). The same occurs in the approximate way to solve Eq. (22) applied up to now when the equation of the form (26) was employed. Of course, this does not pose any problems.

In the general type reaction case the method is applicable in the framework of the approach [12, 13, 14] in which reaction observables are obtained from states $\tilde{\Psi}$ that vanish at large distances like bound states. The approach extensively applied for perturbation induced reactions and proved to be very efficient. Any strong-interaction induced reactions can also be treated in this way. Dynamic equations for states $\tilde{\Psi}$ are of the form of Eq. (22) with $X \rightarrow \tilde{\Psi}$ where q is a given state and the energy E is complex. Because of the latter the solution $\tilde{\Psi}$ is localized and it is a proper object to be found with the help of the expansion over G_{K_0} . The perturbative expansion to calculate $\tilde{\Psi} = \tilde{\Psi}^l + \tilde{\Psi}^h$ is similar to that described above.

The outcome of a calculation are quantities of the form $\Phi(E) = \langle \tilde{\Psi}'(E) | \tilde{\Psi}(E) \rangle$ where $\tilde{\Psi}'$ is a state similar to $\tilde{\Psi}$ for another source term q' . Reaction observables are extracted from $\Phi(E)$ in a simple way as quadratures. When it is sufficient to calculate $\Phi(E)$ only up to the first order in G_{K_0} one need not account for $\tilde{\Psi}^h$ whose contribution in $\Phi(E)$ is of the second order.

III. IMPLEMENTATION

We are dealing with the space of Jacobi coordinates or that of Jacobi momenta and we denote $n = 3A - 3$ the space dimension. Considering matrix elements in the momentum representation we denote $\bar{\pi}$ the n -dimensional momentum vectors and $\Pi = |\bar{\pi}|$ the hyper-momentum. We adopt in (8) $\delta H = \bar{V}(\Pi)$ where $\bar{V}(\Pi)$ is a subsidiary interaction. We have

$$\langle \bar{\pi}' | G_{K_0} | \bar{\pi} \rangle = \frac{\delta^{(n)}(\bar{\pi}' - \bar{\pi}) - [\delta(\Pi' - \Pi)/\Pi^{n-1}] \sum_{K \leq K_0; \nu} Y_{K\nu}^*(\hat{\pi}') Y_{K\nu}(\hat{\pi})}{\Pi^2/(2m) + \bar{V}(\Pi) - E}, \quad (27)$$

$$\langle \bar{\pi}' | U | \bar{\pi} \rangle = \langle \bar{\pi}' | V | \bar{\pi} \rangle - \delta^{(n)}(\bar{\pi}' - \bar{\pi}) \bar{V}(\Pi), \quad (28)$$

where for bound states E means $E_i^{(0)}$. Here $Y_{K\nu}$ is an orthonormalized complete set of hyperspherical harmonics with a given K , and $\hat{\pi}$ denotes a unit vector pointed in the direction of $\bar{\pi}$, $\hat{\pi} = \bar{\pi}/\Pi$, $\hat{\pi}' = \bar{\pi}'/\Pi'$. The hyperangular factor entering (27) may be represented with the simple expression (e.g. [15])

$$\sum_{\nu} Y_{K\nu}^*(\hat{\pi}') Y_{K\nu}(\hat{\pi}) = \frac{K + \frac{n-2}{2}}{2 \cdot \pi^{n/2}} \Gamma\left(\frac{n-2}{2}\right) C_{K^{\frac{n-2}{2}}}(\hat{\pi}' \cdot \hat{\pi}), \quad (29)$$

where $C_K^{\gamma}(x)$ is the Gegenbauer polynomial.

When performing calculations in the coordinate representation we denote ξ the n -dimensional position vectors, $\rho = |\xi|$, and $\hat{\xi} = \xi/\rho$. We have

$$T = T_{\rho} + \frac{\hbar^2}{2M} \frac{\hat{K}^2}{\rho^2}, \quad (30)$$

$$\langle \xi' | T_{\rho} | \xi \rangle = \delta^{(n)}(\xi' - \xi) \left(-\frac{\hbar^2}{2M} \right) \left(\frac{d^2}{d\rho^2} + \frac{n-1}{\rho} \frac{d}{d\rho} \right). \quad (31)$$

Here \hat{K}^2 is the hyperangular momentum operator. In this case we choose:

$$\delta H = \bar{V}(\rho) - T_{\rho} + E, \quad (32)$$

where $\bar{V}(\rho)$ is a subsidiary interaction. It is convenient to represent the corresponding G_{K_0} as a sum of contributions from various K values,

$$G_{K_0} = \sum_{K > K_0} g_K. \quad (33)$$

Then

$$\langle \xi' | g_K | \xi \rangle = \left[\frac{\hbar^2}{2M} \frac{K(K+n-2)}{\rho^2} + \bar{V}(\rho) \right]^{-1} \frac{\delta(\rho' - \rho)}{\rho^{n-1}} \frac{K + \frac{n-2}{2}}{2 \cdot \pi^{n/2}} \Gamma\left(\frac{n-2}{2}\right) C_{K^{\frac{n-2}{2}}}(\hat{\xi}' \cdot \hat{\xi}), \quad (34)$$

$$\langle \xi' | U | \xi \rangle = \langle \xi' | V | \xi \rangle - \delta^{(n)}(\xi' - \xi) [\bar{V}(\rho) - T_\rho + E]. \quad (35)$$

The choice (32) is done to facilitate Monte–Carlo calculations of matrix elements. In the coordinate representation, G_{K_0} that contains the total T as in Eq. (27) would correspond to g_K with hyperradial Green functions varying rapidly at not small K .

When performing calculations it is convenient to include the factor $\sum_\mu |\theta_\mu\rangle \langle \theta_\mu| \equiv I$ in the Green functions, where $\{\theta_\mu\}$ is a complete set of spin–isospin states (c.f. [3]) with simple permutational properties.

The subsidiary interactions $\bar{V}(\rho)$ and $\bar{V}(\Pi)$ can be chosen from the requirement of fast convergence of observables as K_0 increases when one takes into account only, say, the lowest order corrections. Alternatively, one can minimize the ratio of the second order correction to the first order correction for this purpose.

If e.g. in the coordinate representation one writes (suppressing the spin–isospin notation)

$$\Psi_i^{l(m)} = \sum_{K \leq K_0; \nu} \chi_{K\nu}^{(m)}(\rho) Y_{K\nu}(\hat{\xi}) \quad (36)$$

then the equations of Eqs. (11) – (13) and (26) type are of the form

$$-\frac{\hbar^2}{2M} \left(\frac{d^2}{d\rho^2} + \frac{n-1}{\rho} \frac{d}{d\rho} - \frac{K(K+n-2)}{\rho^2} \right) \chi_{K\nu}^{(m)} - E \chi_{K\nu}^{(m)} + \sum_{K'\nu'} (K\nu | V | K'\nu') \chi_{K'\nu'}^{(m)} = (K\nu | q), \quad (37)$$

where $K \leq K_0$ and $K' \leq K_0$.

Their right–hand sides as well as the $\Psi^{h(m)}$ components are to be calculated with the Monte–Carlo method. Integrands depend on high K values only via Gegenbauer polynomials (29) entering (27) and (34). While these polynomials are rather quickly oscillating all other factors in the integrands are smooth functions of coordinates or momenta. In the case $m = 1$ one may simplify a calculation taking the argument of Gegenbauer polynomials as a new integration variable. Integration over this variable may be done with the help of the regular Gauss–Gegenbauer quadratures while integration over other variables that are smooth may be done with the Monte–Carlo method. A suitable change of variables is described in Appendix. This can also be done in the case of a momentum representation calculation. At the same time there are indications (e.g. [16]) that direct Monte–Carlo integration may be suitable even at rather large K values.

Local components are dominating components of nuclear forces derived from the effective

field theory. If $m \geq 2$ correction terms are retained in a calculation a reasonable simplification may be to account for only those local components in these terms.

Direct solution of Eqs. (37) in the form they are written down is hampered by the large centrifugal barriers $K(K + n - 2)/\rho^2$. In the bound-state case a practical procedure is to expand $\chi_{K\nu}^{(m)}(\rho)$ over a set of functions that reduces Eqs. (37) to linear equations. Such linear equation sets of a large size may efficiently be solved with a version of the method of Ref. [1], i.e. using an expansion over another parameter of the K_0^{-2} type. For complex E values with positive real parts and rather small imaginary parts entering Eqs. (37) in the reaction calculations convergence of the expansion procedure to solve Eqs. (37) is slow. And for real E values entering Eqs. (37) in the other type calculations of reactions this procedure may lead to unphysical singularities in reaction observables. Other efficient solution methods are available for this purpose.

Integral transforms $\Phi(E)$ are required at sufficiently many values of complex energies E to perform a satisfactory inversion [12, 13, 14]. But one need not solve Eqs. (37) for all these E values. A better approach is to solve these equations for a rather small number of E values and to obtain $\Phi(E)$ for a larger set of E values via interpolation. The transforms $\Phi(E)$ are smooth functions and this procedure is safe and accurate.

IV. ESTIMATES

First let us consider numerical estimates for the ${}^4\text{He}$ system in the 0^+ state. In [3] the accurate α -particle binding energy pertaining to a proper subset of hyperspherical harmonics has been compared with energies calculated approximately as follows. Only matrix elements of NN force $(K\nu|V|K'\nu')$ such that either K or K' does not exceed some K_0 were retained in the system of equations. This approximation is equivalent to the approximation $\Gamma \rightarrow G_{K_0}$ in Eq. (9). The approximate solution thus obtained is close to that given by the $m = 1$ approximation of the method described above. It accounts for corrections to binding energy up to $m = 3$. The corresponding approximate energy values as a function of K_0 along with the accurate value are shown in Table 4 of Ref. [3].

At $K_0 = 14$ the difference between the two values equals to 0.23 MeV for the NN interaction with a very strong repulsive core. In this connection one needs to remember that the binding energy considered is a small difference between two large quantities, potential and

kinetic energy, which deteriorates the accuracy. This comparison refers to the case when the above mentioned mean field \bar{V} is set to zero. Inclusion of the mean field in the calculation would improve the convergence. Furthermore, it is seen from Table 4 of [3] that for another NN force having a softer core convergence with respect to K_0 is much faster. One may note in this connection that nuclear forces derived from the effective field theory are much softer than that used in the above comparison. For such forces convergence with respect to a maximum K value retained in a calculation is considerably faster than for phenomenological local realistic forces [17]. Naturally, for effective field theory forces one may expect faster convergence as to K_0 as well.

Now let us perform an estimate for the case of reactions in the same system. We consider the above outlined approach dealing with complex energy E , and we set $E = \sigma_R + i\sigma_I$. We take $\sigma_I = 10$ MeV which is a good value for e.g. electromagnetic processes [14]. We consider again the case $K_0 = 14$ and again we shall not include the mean field \bar{V} . We employ the so called AV4 potential that is the central component of the realistic AV18 NN interaction [18]. For the estimate purposes we adopt the following model. We represent the subspace with $K \leq K_0$ with a single hyperspherical harmonic with $K = 0$. We represent the subspace with $K > K_0$ with a single hyperspherical space-symmetric "potential" harmonic with $K = 16$. Thus in our model we deal with two coupled differential equations that have the form of Eqs. (37) and that correspond to the hyperspherical expansion of the state $\tilde{\Psi}$ described in the preceding section. The right-hand side source term in the first of the equations was set to be $\exp(-\rho/0.4 \text{ fm})$. The form of the source terms is not very important for the estimate and in the second of the equations the source term was set to zero. In general, at large ρ values the state $\tilde{\Psi}$ contains the 3N+N cluster components, in particular. When K is less or about $\rho/\sqrt{3}R$, R being the range of the 3N cluster, in equations of the type we consider matrix elements may produce coherent effects. Then it would not be suitable for our estimates to represent the $K > K_0$ subspace with a single hyperspherical harmonics. In our case, however, a typical extension in ρ of the state $\tilde{\Psi}$ is 10 fm, c.f. below, and with our K_0 value these effects are not relevant. The results for the quantity of interest $\Phi(E) = \langle \tilde{\Psi}(E) | \tilde{\Psi}(E) \rangle$ are shown in Fig. 1. The curves labeled as the first and the second approximation correspond, respectively, to calculations in the frameworks of $m = 1$ and $m = 2$ approximations as described in the preceding section. We also note that the central force we use for the estimate has a strong repulsive core of the height of 2.7 GeV. As above,

⁴He, AV4 interaction, "K₀=14"

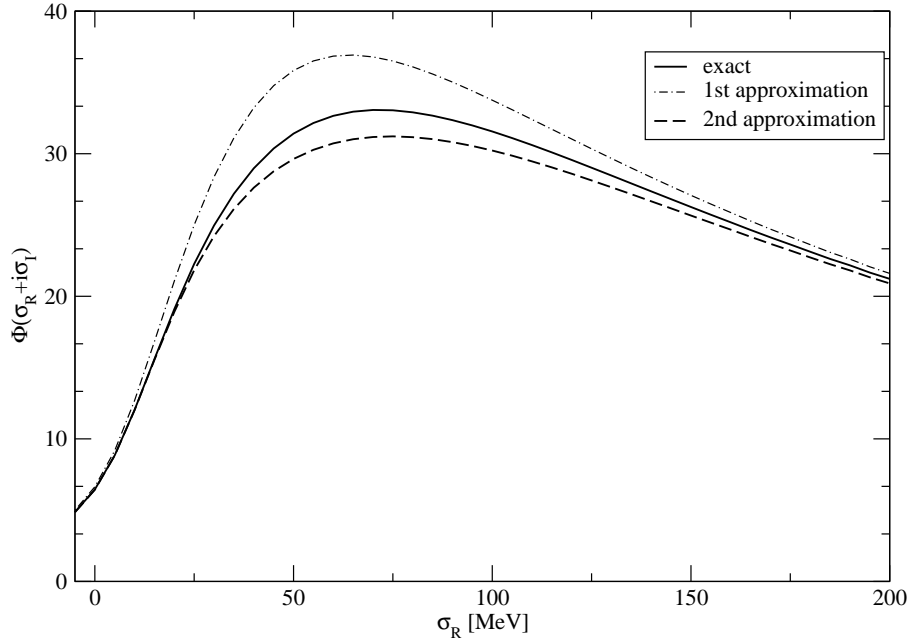


FIG. 1: Integral transform of the form factor of the type of those governing reaction amplitudes or representing response functions. The calculation is done for the model described in the text. The quantity σ_I was set to be 10 MeV. Exact values and the $m=1$ and $m=2$ approximations are compared.

one may note that use of a nuclear force derived from the effective field theory and inclusion of the mean field \bar{V} in a calculation would improve the rate of convergence.

In the problems we considered there exists several hundreds hyperspherical harmonics with $K \leq 14$ that is acceptable. A swift rise of the number of hyperspherical harmonics with the same K starts at K values about 10, see Table 3 from [3]. So it would be desirable to keep K_0 at the level of 10–12 in a calculation.

In general, conditions for applicability of the perturbation approach we consider are

$$||G_{K_0}U\Psi^h|| \ll ||\Psi^h||, \quad ||G_{K_0}V\Psi^l|| \ll ||\Psi^l||. \quad (38)$$

A better accuracy in the first of these conditions is provided reducing U via inclusion of the mean field \bar{V} . The second of these conditions contains the interaction in the form $Q_{K_0}VP_{K_0}$. It is non-diagonal in hyperspherical quantum numbers and is substantially "smaller" than

$Q_{K_0} V Q_{K_0}$ for this reason. For bound states and for scattering states in the most interesting case when $\text{Re}E$ is smaller or about $|\langle V - \bar{V} \rangle|$ the first of these conditions, for example, may roughly be represented as

$$\frac{\hbar^2}{2m\bar{\rho}^2} \left(K_0 + \frac{n-2}{2} \right)^2 \gg |\langle V - \bar{V} \rangle|, \quad (39)$$

where $\bar{\rho} \simeq A^{1/2} \langle r^2 \rangle^{1/2}$ is the corresponding range either of the bound state Ψ , or that of the component X of the continuum state above, or that of the state $\tilde{\Psi}$ above while $\langle V - \bar{V} \rangle$ is the average value over Ψ , or X , or $\tilde{\Psi}$. [21] In the case, for example, of the ${}^7\text{Li}$ two-cluster ground state the rate of convergence may be increased if one diminishes $\bar{\rho}$ separating out the cluster component of the state (c.f. Eq. (21).

The value of $\bar{\rho}$ in (39) in the case of the $\tilde{\Psi}$ state is about $(\text{Im}k)^{-1}$ where $(\hbar k)^2/(2M) = E$. If $E = \sigma_R + i\sigma_I$ then $\text{Im}k = 2^{-1/2}[(\sigma_R^2 + \sigma_I^2)^{1/2} - \sigma_R]^{1/2}$. The choice of σ_I is discussed in [14], see also [19].

It occurs that when $\text{Re}E = \sigma_R$ increases at given K_0 the left-hand sides in Eqs. (38) decrease if G_{K_0} depends on E as in Eq. (27).

We note that the separation out of two-cluster components to diminish the $\bar{\rho}$ value may also be done when calculating the $\tilde{\Psi}$ quantities. At large distances the relative motion factor is an outgoing (neutral or Coulomb) wave with a complex wave number having a positive imaginary part in this case. This ensures vanishing of these components at large distances.

In the $n+n+p$ continuum-state case the method may be applied also above the three-body breakup threshold despite the fact that the component X is not localized. The reason is that average potential energy for the total breakup channel decreases with ρ as ρ^{-3} . But in the 3-body case it may be profitable to do the whole calculation in the matrix form i.e. to use the HH expansion for the perturbative calculation also of the $K > K_0$ contributions.

In conclusion, an area for testing realistic nuclear forces may be substantially extended with the help of the presented approach. For this purpose, it is required to investigate the issue of Monte-Carlo computing the perturbative corrections.

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APPENDIX

When one takes $\hat{\xi}' \cdot \hat{\xi}$ as a new integration variable one needs to define the whole set of integration variables in a way that the integrand remains non-singular. This can be done e.g. as follows. Let us express a unit vector $\hat{\xi} = \{\hat{\xi}_1, \dots, \hat{\xi}_n\}$ in terms of another unit vector $\hat{\eta}$,

$$\hat{\xi}_i = \sum_{j=1}^n g_{ij} \hat{\eta}_j,$$

where g_{ij} is an orthogonal matrix such that its first column is $g_{i1} = \hat{\xi}'_i$ and g_{ij} is arbitrary otherwise. One then has $\hat{\xi}' \cdot \hat{\xi} = \sum_{i,j} g_{i1} g_{ij} \hat{\eta}_j = \hat{\eta}_1$. Let us parametrize the components of $\hat{\eta}$ as follows,

$$\hat{\eta}_1 = \cos \varphi, \quad \hat{\eta}_j = \hat{v}_{j-1} \sin \varphi, \quad j = 2, \dots, n,$$

where \hat{v}_i are components of a unit vector \hat{v} on a hypersphere in a $n-1$ -dimensional subspace. Taking into account that

$$d\hat{\xi} = d\hat{\eta} \equiv (\sin \varphi)^{n-2} d\hat{v} d\varphi$$

one then may rewrite e.g. integrals of the structure $\langle F_1 | G_{K_0} | F_2 \rangle$ as

$$\sum_{K > K_0} \frac{K + \frac{n-2}{2}}{2 \cdot \pi^{n/2}} \Gamma\left(\frac{n-2}{2}\right) \int \rho^{n-1} d\rho (\sin \varphi)^{n-2} d\varphi d\hat{v} d\hat{\xi}' F_1^*(\rho \hat{\xi}) \left[\frac{\hbar^2}{2m} \frac{K(K+n-2)}{\rho^2} + \bar{V}(\rho) \right]^{-1} C_{K^{\frac{n-2}{2}}}(\cos \varphi) F_2(\rho \hat{\xi}'),$$

where the components of the n -dimensional unit vector $\hat{\xi}$ entering F_1 are parametrized as follows,

$$\hat{\xi}_i = \hat{\xi}'_i \cos \varphi + \left(\sum_{j=2}^n g_{ij}(\hat{\xi}') \hat{v}_{j-1} \right) \sin \varphi.$$

The integrations over $d\rho$, $d\hat{\xi}'$, and $d\hat{v}$ may be performed with the Monte-Carlo method while the remaining integration over $d\varphi$ may be done with the help of regular quadratures. Integrals at a given ρ over the hypersphere which pertain to the $m = 1$ correction are transformed similarly.

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- [21] If one wants to derive Eq. (39) with use of the momentum representation one needs to take into

account that if a coordinate representation wave function $f(\xi)$ is localized within a hyperradius ρ then the momentum representation quantities $(Y_{K\nu}(\hat{\pi})|f(\pi))$ are very small at Π values such that $\Pi\rho \ll K + (n - 2)/2$.